# Formal exponential maps and Fedosov resolutions

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- Classical exponential map:
  - Lie theory: $\exp : \mathfrak{g} \to G : A \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} A^n$
  - smooth manifold: $\exp^{\nabla} T_M|_p \to M : \exp^{\nabla}(v_p) =$ the value of the geodesic along $v_p$ at time 1
- Laurent-Gengoux, Stiénon, Xu (arXiv:1408.2903, Adv Math 2021): found an iteration formula of the infinity jet of exp map of Lie groupoid. Above 2 classical exp maps are special cases.
- Stiénon and I (arXiv:1605.09722, IMRN 2019):
  - used a similar iteration formula to define formal exponential map $\mathrm{pbw} = \mathrm{pbw}^{\nabla}$ for ($\mathbb{Z}$-)graded manifolds
  - extended the classical construction of Dolgushev-Fedosov resolution to graded manifolds (important in deformation quantization)
  - used pbw to give a new construction of this resolution
- Seol, Stiénon, Xu (arXiv:2106.00812, to appear in Communications in Mathematical Physics): studied pbw for dg manifolds (obstruction of compatibility = Atiyah class).

1. Infinite jet of exponential map
2. Formal exponential map for graded manifolds
3. Application to Fedosov resolutions

**Theorem (Poincaré–Birkhoff–Witt)**

*Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The map*

$$\mathrm{pbw} : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$$

*defined by the explicit formula*

$$\mathrm{pbw}(X_1 \odot \cdots \odot X_n) = \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

*and is an isomorphism of coalgebras.*

$$\mathfrak{g} \xrightarrow{\exp} G$$

$$S(\mathfrak{g}) \cong \mathcal{D}_0(\mathfrak{g}) \xrightarrow{\exp_* = \mathrm{pbw}} \mathcal{D}_e(G) \cong \mathcal{U}(\mathfrak{g})$$

# Exponential map for smooth manifolds

- connection $\nabla$ on smooth manifold $M$
- $\exp : T_M \to M \times M$ (bundle map) defined by geodesics.
  - $\Gamma(S(T_M))$ seen as space of differential operators on $T_M$, all derivatives in the direction of the fibers, evaluated along the zero section of $T_M$
  - $\mathcal{D}(M)$ seen as space of differential operators on $M \times M$, all derivatives in the direction of the fibers, evaluated along the diagonal section $M \to M \times M$
- $\mathrm{pbw} := \exp_* : \Gamma(S(T_M)) \xrightarrow{\cong} \mathcal{D}(M)$ is an isomorphism of left modules over $C^\infty(M)$ called Poincaré–Birkhoff–Witt isomorphism

Taylor series of $T_m M \xrightarrow{\exp} \{m\} \times M \xrightarrow{f} \mathbb{R}$ at $0_m \in T_m M$ is

$$\sum_{J \in \mathbb{N}_0^n} \frac{1}{J!} \big(\mathrm{pbw}(\partial_x^J) f\big)(m) \otimes y^J \qquad \in \widehat{S}(T_m^\vee M).$$

# Algebraic characterization of pbw

## Theorem (Laurent-Gengoux, Stiénon, Xu)

$$\mathrm{pbw}(f) = f, \quad \forall f \in C^\infty(M);$$
$$\mathrm{pbw}(X) = X, \quad \forall X \in \mathfrak{X}(M);$$
$$\mathrm{pbw}(X^{n+1}) = X \cdot \mathrm{pbw}(X^n) - \mathrm{pbw}(\nabla_X X^n), \quad \forall n \in \mathbb{N}.$$

Therefore, for all $n \in \mathbb{N}$ and $X_0, \dots, X_n \in \mathfrak{X}(M)$,

$$\mathrm{pbw}(X_0 \odot \cdots \odot X_n) = \frac{1}{n+1} \sum_{k=0}^{n} \left\{ X_k \cdot \mathrm{pbw}(X^{\{k\}}) - \mathrm{pbw}\left(\nabla_{X_k}(X^{\{k\}})\right) \right\}$$

where $X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$.

## Proposition

$\mathrm{pbw} : \Gamma(S(T_M)) \to \mathcal{D}(M)$ *is an isomorphism of coalgebras over* $C^\infty(M)$.

# Differential graded manifolds

**Definition**

A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with support $M$ is a sheaf $\mathcal{R}$ of $\mathbb{Z}$-graded commutative algebras over $M$ such that $\mathcal{R}(U) \cong C^\infty(U) \otimes S(V^\vee)$ for sufficiently small open subsets $U$ of $M$ and some $\mathbb{Z}$-graded vector space $V$. Here $S(V^\vee)$ denotes the graded algebra of polynomials on $V$.

$$C^\infty(\mathcal{M}) := \mathcal{R}(\mathcal{M})$$

**Definition**

A dg manifold is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ endowed with a vector field $Q \in \mathfrak{X}(\mathcal{M})$ of degree $+1$ such that $[Q, Q] = 2\, Q \circ Q = 0$.

# Why dg manifolds?

- The theory of super manifolds ($\mathbb{Z}_2$-graded manifolds) was motivated by the theory of fermions and bosons.
- AKSZ (Alexandrov-Kontsevich-Schwarz-Zaboronsky) formulation of quantum field theory was based on dg manifolds. ($\mathbb{Z}_2$-graded.)
- Many interesting examples: Lie algebras, complex manifolds, homotopy Lie algebras, foliations, higher Lie algebroids, etc.
- Dg manifolds (of positive amplitudes) is a model for "derived differential geometry."

# Formal exponential map

- Connections can be defined on graded manifolds by adding suitable signs.
- Difficult to study geodesics on graded manifolds. It is complicated to generalize the classical exponential map.
- A short cut: The algebraic relations satisfied by pbw serve as an alternative definition.

The formal exponential map associated to a connection $\nabla$ on $T_{\mathcal{M}}$ is the morphism of left $C^{\infty}(\mathcal{M})$-modules

$$\mathrm{pbw} : \Gamma(S(T_{\mathcal{M}})) \to \mathcal{D}(\mathcal{M}),$$

inductively defined by the relations

$$\mathrm{pbw}(f) = f \qquad \forall f \in C^{\infty}(\mathcal{M}),$$
$$\mathrm{pbw}(X) = X \qquad \forall X \in \Gamma(T_{\mathcal{M}}),$$
$$\mathrm{pbw}(X_0 \odot \cdots \odot X_n) = \frac{1}{n+1} \sum_{k=0}^{n} \epsilon_k \left\{ X_k \cdot \mathrm{pbw}(X^{\{k\}}) - \mathrm{pbw}\left(\nabla_{X_k} X^{\{k\}}\right) \right\}$$

for $n \in \mathbb{N}$ and homogeneous $X_0, \ldots, X_n \in \Gamma(T_{\mathcal{M}})$,
$\epsilon_k = (-1)^{|X_k|(|X_0|+\cdots+|X_{k-1}|)}$,
$X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$.

**Theorem (L, Stiénon)**

*The formal exponential map*

$$\mathrm{pbw} : \Gamma(S^{\leqslant k}(T_{\mathcal{M}})) \to \mathcal{D}^{\leqslant k}(\mathcal{M})$$

*is a well-defined isomorphism of filtered coalgebras over* $C^\infty(\mathcal{M})$.

**Remark (Seol-Stiénon-Xu 2021)**

*An analogous isomorphism exists on a dg manifold iff the Atiyah class vanishes.*

1. Infinite jet of exponential map

2. Formal exponential map for graded manifolds

3. Application to Fedosov resolutions

- Fedosov constructed an operator for solving the deformation quantization problem on symplectic manifolds. (J. Differential Geom. 1994)
- Emmrich and Weinstein studied Fedosov-type operators for smooth manifolds. The construction was given by iteration method. (Lie Theory and Geometry, 1994)
- In the same paper, Emmrich and Weinstein revealed a relationship between exponential maps and Fedosov-type operators. Their method was based on the geodesic equations.
- Dolgushev realized that Emmrich-Weinstein construction gives rise to resolutions of the algebra of smooth function on a manifold and applied it to globalizing Kontsevich formality morphism. (Adv. Math. 2005)

Stiénon and I (IMRN 2019) extended Dolgushev's construction and got Dolgushev-Fedosov resolutions for graded manifolds:

$$C^\infty(\mathcal{M}) \xhookrightarrow{\tau^\nabla} \Omega^0(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{D^\nabla} \Omega^1(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{D^\nabla} \cdots$$

We apply the formal exponential map pbw to

- obtain a new construction of Dolgushev-Fedosov resolutions
- prove Emmrich–Weinstein type theorem for graded manifolds.

# Classical construction

On a graded manifold $\mathcal{M}$ of dimension $n$, one has the

- Koszul operator $\delta : \Omega^p(\mathcal{M}, S^q T^\vee_{\mathcal{M}}) \to \Omega^{p+1}(\mathcal{M}, S^{q-1} T^\vee_{\mathcal{M}})$,

$$\delta(\omega \otimes f) = \sum_{k=1}^{n} (-1)^{\left|\frac{\partial}{\partial y_k}\right||\omega|} dx_k \wedge \omega \otimes \frac{\partial}{\partial y_k}(f)$$

- homotopy operator
$h : \Omega^p(\mathcal{M}, S^q T^\vee_{\mathcal{M}}) \to \Omega^{p-1}(\mathcal{M}, S^{q+1} T^\vee_{\mathcal{M}})$,

$$h(\omega \otimes f) = \frac{1}{p+q} \sum_{k=1}^{n} (-1)^{|y_k||\omega|} i_{\frac{\partial}{\partial x_k}} \omega \otimes y_k \cdot f.$$

- Twist $-\delta$ by covariant derivative $d^\nabla$, but $(-\delta + d^\nabla)^2 \neq 0$.
- Consider operators of the form

$$D^\nabla = -\delta + d^\nabla + X^\nabla$$

where the correction

$$X^\nabla \in \Omega^1(\mathcal{M}, \hat{S}^{\geqslant 2}(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}}).$$

- Solve $X^\nabla$ by the equation $(D^\nabla)^2 = 0$.

**Theorem**

*Let $\nabla$ be a torsion-free connection. There exists a unique degree one element $X^\nabla \in \Omega^1(\mathcal{M}, \hat{S}^{\geqslant 2}(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}})$ (given by iterating an equation) such that*

- $h(X^\nabla) = 0$, *(more precisely, $h$ is $h \otimes \mathrm{id}$)*
- $(D^\nabla)^2 = 0$.

# Our construction

- $\mathcal{M}$ graded manifold of dimension $n$
- Pick a connection $\nabla$ on $T_{\mathcal{M}}$.
- Obtain associated $\mathrm{pbw} : \Gamma(S(T_{\mathcal{M}})) \to \mathcal{D}(\mathcal{M})$.
- Consider the connection $\nabla^{\lightning}$ on $S(T_{\mathcal{M}})$ defined by

$$\nabla^{\lightning}_X S := \mathrm{pbw}^{-1}\left(X \cdot \mathrm{pbw}(S)\right)$$

  for all $X \in \Gamma(T_{\mathcal{M}})$ and $S \in \Gamma(S(T_{\mathcal{M}}))$.
- The connection $\nabla^{\lightning}$ induces a connection on the dual bundle $\hat{S}(T^{\vee}_{\mathcal{M}})$. By abusing notations, we use the same symbol $\nabla^{\lightning}$ for this induced connection.
- We have the covariant derivative

$$d^{\nabla^{\lightning}} : \Omega^p(\mathcal{M}, \hat{S}(T^{\vee}_{\mathcal{M}})) \to \Omega^{p+1}(\mathcal{M}, \hat{S}(T^{\vee}_{\mathcal{M}}))$$

## Theorem (L, Stiénon)

- *The connection $\nabla^{\natural}$ is flat. Namely*

$$(d^{\nabla^{\natural}})^2 = 0$$

- *If $\nabla$ is a torsion-free connection, then*

$$d^{\nabla^{\natural}} = -\delta + d^{\nabla} + X^{\nabla}$$

*with*

$$X^{\nabla} \in \Omega^1(\mathcal{M}, \hat{S}^{\geqslant 2}(T^{\vee}_{\mathcal{M}}) \otimes T_{\mathcal{M}}),$$

$\mathrm{degree}(X^{\nabla}) = +1$, *and* $h(X^{\nabla}) = 0$.

*Therefore,*

$$d^{\nabla^{\natural}} = D^{\nabla}$$

# Emmrich–Weinstein theorem for graded manifolds

## Theorem (L, Stiénon)

*There is a degree zero injective algebraic homomorphism $\tau^\nabla$ which defines a resolution for $C^\infty(\mathcal{M})$*

$$\tau^\nabla : C^\infty(\mathcal{M}) \to (\Omega^\bullet(\mathcal{M}, \hat{S}(T^\vee_\mathcal{M})),\ d^{\nabla^\natural})$$

*Furthermore, the map $\tau^\nabla$ can be expressed by*

$$\tau^\nabla(f) = \sum_{J\in\mathbb{N}_0^n} \frac{1}{J!} y^J \Big( \operatorname{pbw}\left(\overleftarrow{\partial_x^J}\right) f \Big) \qquad \forall f \in C^\infty(\mathcal{M}).$$

$$\overleftarrow{\partial_x^J} = \underbrace{\partial_{x_n} \odot \cdots \odot \partial_{x_n}}_{J_n \text{ factors}} \odot \underbrace{\partial_{x_{n-1}} \odot \cdots \odot \partial_{x_{n-1}}}_{J_{n-1} \text{ factors}} \odot \cdots \odot \underbrace{\partial_{x_1} \odot \cdots \odot \partial_{x_1}}_{J_1 \text{ factors}}$$

and $y^J = y_1^{J_1} y_2^{J_2} \cdots y_n^{J_n}$ for $J = (J_1, J_2, \ldots, J_n) \in \mathbb{N}_0^n$

## Corollary (Emmrich, Weinstein)

*For classical (i.e. non-graded) manifolds,*

$$\tau^{\nabla}(f) = \sum_{J \in \mathbb{N}_0^n} \frac{1}{J!}\big(\mathrm{pbw}(\partial_x^J) f\big) \otimes y^J \qquad \forall f \in C^\infty(M).$$

*Thus,* $\tau^{\nabla}(f) =$ *Taylor expansion of* $\exp^* f$ *along zero section.*

# Thank you!

- Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu, *Poincaré-Birkhoff-Witt isomorphisms and Kapranov dg-manifolds*, Adv. Math. **387** (2021), Paper No. 107792, 62. MR 4271478
- Hsuan-Yi Liao and Mathieu Stiénon, *Formal exponential map for graded manifolds*, Int. Math. Res. Not. IMRN (2019), no. 3, 700–730. MR 3910470
- Seokbong Seol, Mathieu Stiénon, and Ping Xu, *Dg manifolds, formal exponential maps and homotopy Lie algebras*, arXiv e-prints (2021), arXiv:2106.00812.